Quantum Field Theory: Exercise Session 5 26 June 2012

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Problem 1: Decay of a scalar particle

Consider the following Lagrangian, involving two real scalar fields, Φ and φ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi\, \partial^\mu \Phi - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \frac{1}{2}m^2\varphi^2 - \mu\, \Phi\varphi\varphi\,. \tag{1}$$

The last term is an interaction term, with coupling constant μ, which allows the particle Φ to decay into two φ's.

(a) Write down the momentum space Feynman rules for this theory, and hence the Feynman diagram for the decay of Φ to lowest order in μ.

(b) Obtain the invariant matrix element $\mathcal{M}$, defined from the scattering matrix $S = 1+iT$ by:

$$\langle k_{\varphi_1} k_{\varphi_2} | iT | k_\Phi \rangle = (2\pi)^4\, \delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2})\, i\mathcal{M}(k_\Phi \to k_{\varphi_1}, k_{\varphi_2})\,, \tag{2}$$

and given diagrammatically by:

$$i\mathcal{M} = \{\text{the sum of all connected amputated Feyman diagrams.}\} \tag{3}$$

(c) Compute the decay rate of the Φ particles in their rest frame, to lowest order in μ, using the relation:

$$\Gamma = \frac{1}{2M} \prod_{f=\varphi_1,\varphi_2} \int d\tilde{k}_f\, |\mathcal{M}(k_\Phi \to k_{\varphi_1}, k_{\varphi_2})|^2\, (2\pi)^4\, \delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2})\,. \tag{4}$$

Hint: A change of variables $\delta(f(x))dx = \frac{dx}{dy}\delta(y)dy$ with $y = f(x)$ shows that:

$$\frac{\delta\left(\sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M\right)}{\sqrt{|\vec{k}|^2 + m_1^2}\sqrt{|\vec{k}|^2 + m_2^2}} |\vec{k}|^2 d|\vec{k}| = \frac{|\vec{k}|}{M}\delta(y)dy \tag{5}$$

with $y = \sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M$.

(d) Compute the lifetime $\tau = \Gamma^{-1}$ of Φ to lowest order in μ.

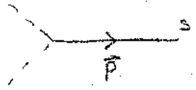
(e) What is the lower bound for M in order for a decay to be possible?

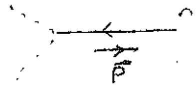
Problem 2: Decay of the charged pion

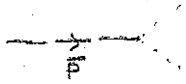
The negatively charged pion can be represented by a complex scalar field φ, the muon by a Dirac field ψ and the muon neutrino by a helicity-projected left-handed Dirac field $(1-\gamma^5)\chi$. The pion decays almost exclusively into a muon and muon antineutrino, and this process can be described by a Lagrangian density:

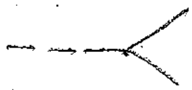
$$\begin{aligned} \mathcal{L} = \quad & \partial_\nu\varphi^*\partial^\nu\varphi - m_\pi^2|\varphi|^2 + \bar{\psi}(i\partial\!\!\!/ - m_\mu)\psi + \bar{\chi}i\partial\!\!\!/\chi \\ & + \frac{G}{\sqrt{2}} f_\pi \left[\partial_\nu\varphi\bar{\psi}\gamma^\nu(1-\gamma^5)\chi + h.c.\right] \end{aligned} \tag{6}$$

where G and f_π are coupling constants. The momentum space Feyman rules for this theory read:

$\langle\vec{p}, s|\bar{\psi} \;=$  $= \bar{u}^s(p)$ fermion (7)

$\langle\vec{p}, r|\chi \;=$  $= v^r(p)$ antifermion (8)

$\varphi|\vec{p}\rangle \;=$  $= 1$ (9)

 $= \frac{G}{\sqrt{2}} f_\pi\, p_\nu \gamma^\nu (1-\gamma^5)\,,$ (10)

(a) Compute the lifetime of the pions.

Hint: When computing the mod squared of the invariant matrix element, $|\mathcal{M}|^2$, you should sum over all possible final spin states, using the completeness relations:

$$\sum_{s=\pm} u_a^s(p)\bar{u}_b^s(p) = (p\!\!\!/ + m_\mu)_{ab} \quad \text{where} \quad (p\!\!\!/ - m_\mu)u^s(p) = 0\,, \tag{11}$$

$$\sum_{r=\pm} v_a^r(p)\bar{v}_b^r(p) = p\!\!\!/_{ab} \quad \text{where} \quad p\!\!\!/\, v^r(p) = 0\,. \tag{12}$$

The gamma-trace identities are:

$$\mathrm{Tr}(\text{any odd number of } \gamma\text{'s}) = 0 \tag{13}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \tag{14}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0 \tag{15}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{16}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{17}$$

(b) Why is the process $\pi^- \to e^- + \bar{\nu}_e$ suppressed?